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Non-self-similar collapsing solutions of the nonlinear Schrödinger equation at the critical dimension

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The dynamical problem of a spherically symmetric wave collapse is investigated in the framework of the nonlinear Schrödinger equation defined at the critical dimension. Collapsing solutions are shown to remain self-similar for spatial coordinates below a cutoff radius only, and to exhibit at larger distances a non-self-similar tail whose expression is explicitly computed. A rapid method used to study the time behavior and the stability of the contraction rate associated with these singular solutions is also derived.

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The collapse of strongly nonlinear wave packets has been the topic of intense investigations in various areas of physics [1–3]; this phenomenon consists in the self-focusing and the ultimate blowup of multidimensional localized structures whenever their initial mass exceeds a critical value. Such a singular behavior particularly appears within the solutions of the nonlinear Schrödinger equation

$$i\frac{\partial}{\partial t}\psi+r^{1-d}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r}\psi+|\psi|^{4/d}\psi=0 , \tag{1}$$

where $\psi(r,t)$ represents a scalar wave field, which will be regarded as radially symmetric in a d-dimensional space. The system (1) admits two main invariants, namely, the mass integral $N\{\psi\}\equiv\int_0^\infty|\psi|^2r^{d-1}dr$ and the Hamiltonian H, given by

$$H\{\psi\}\equiv\int_0^\infty\left\{|\nabla\psi|^2-\frac{d}{d+2}|\psi|^{2(d+2)/d}\right\}r^{d-1}dr . \tag{2}$$

Equation (1) has well-known stationary solutions [1,4] of the form $\psi(r,t)=e^{it}R_0(r)$, where $R_0(r)$ is the real eigenfunction of Eq. (1) satisfying the differential equation

$$\left[-1+r^{1-d}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r}+R_0^{4/d}\right]R_0=0 , \tag{3}$$

with the two boundary conditions $R_0(r\to\infty)\to 0$ and $\partial R_0(0)/\partial r=0$. From relations (2) and (3), it follows that the soliton-type function R_0 verifies $H(R_0)=0$ [4].

The collapsing states of Eq. (1) exhibit a diverging amplitude and a radial size $f(t)$ that decreases to zero as t tends to the henceforth called "collapse time" t_*. Such collapsing states are described in terms of solutions of the following form:

$$\psi(r,t)=g(\tau)^{-d/2}\phi(\xi,\tau)\exp[i\tau-ia(\tau)\xi^2/4] . \qquad \arg\phi(0,\tau)=0 , \tag{4}$$

in which the new space and time coordinates are defined by $\xi=r/f(t)=r/g(\tau)$ and $\tau=\int_0^t f^{-2}(u)du$, where the contraction rate $g(\tau)$ and the time τ must tend respectively to zero and to infinity as $t\to t_*$; $a(\tau)$ denotes the *positive* function $a(\tau)\equiv -f_t f=-g_\tau/g$. The substitution (4) ensures the mass invariance, namely, $N\{\psi\}=N\{\phi\}\equiv\int_0^\infty|\phi|^2\xi^{d-1}d\xi$, and the equation of evolution for ϕ is

$$i\frac{\partial}{\partial\tau}\phi+\xi^{1-d}\frac{\partial}{\partial\xi}\xi^{d-1}\frac{\partial}{\partial\xi}\phi+|\phi|^{4/d}\phi+(\varepsilon\xi^2-1)\phi=0 , \tag{5}$$

where $\varepsilon(\tau)$ is the function $\varepsilon(\tau)\equiv -f^3f_{tt}/4=(a^2+a_\tau)/4$. The problem of the critical collapse therefore reduces to the nonlinear eigenvalue problem consisting in determining the function $\varepsilon(\tau)$ in such a way that the solution $\phi(\xi,\tau)$ satisfies the boundary conditions $\phi(\xi,\tau)\to 0$ and $\partial_\xi\phi(\xi,\tau)\to 0$ for $\xi\to\infty$; the contraction rate $f(t)$ may in turn be deduced from the differential equation $f^3f_{tt}=-4\varepsilon$. Exactly self-similar solutions correspond to functions $\phi(\xi,\tau)$ which no longer depend on τ for $\tau\to\infty$, which imposes $\varepsilon(\tau)\to$const as $\tau\to\infty$. On the other hand, it has been numerically observed that the solution admits in this latter limit an exact self-similar core of the form $\psi\to g^{-d/2}R_0(\xi)\exp(i\tau)$ within a bounded domain in ξ. From this result it follows that the characteristic behavior $\varepsilon(\tau)\to 0$ must hold for $\tau\to\infty$ [ensuring then $a(\tau)\to 0$ with $|a_\tau|\ll a^2$ in the same limit]. Among the numerous contraction rates proposed in the literature [5–12], the double-logarithmic scaling law

$$f(t)=\left[\frac{2\pi(t_*-t)}{\ln\ln\left[\frac{1}{t_*-t}\right]}\right]^{1/2} \tag{6}$$

was confirmed by many authors by means of various perturbation methods and verified by accurate numerical computations [7–13]. The analytical procedures used in order to derive (6) are based on an expansion of the solution $\phi(\xi,\tau)$ into a central nonlinear core and an asymptotic tail vanishing for $\varepsilon\to 0$. In this limit, this solution ϕ is standardly approximated by a so-called "quasi-self-similar" state for which the function $\varepsilon(\tau)$ varies sufficiently slowly for the approximation $\partial\phi/\partial\tau=0$ to be valid. Within this approximation, Eq. (5) formally differs from Eq. (3) by the extra contribution $\varepsilon\xi^2$ introduced by the substitution (4). One may define accordingly the turning point ξ_T as $\xi_T\equiv 1/\sqrt{\varepsilon}$; as the latter goes to infinity for $\tau\to\infty$, it lies in the spatial range where the nonlinear term $|\phi|^{4/d}$ can be neglected. Since Eq. (5)

reduces to Eq. (3) in the domain $\xi < \xi_T$, ϕ is expected to be close to the solution R_0 in this range; conversely, ϕ will significantly differ from R_0 in the opposite domain. For this reason the quasi-self-similar solution is sought under the form $\phi(\xi,\varepsilon)=\phi_0(\xi,\varepsilon)+\phi_T(\xi,\varepsilon)$, where ϕ_0 represents the central core of the solution that tends towards the exactly self-similar state R_0 as $\varepsilon \to 0$, and dominates over ϕ_T in the range $\xi < \xi_T$. In the opposite domain $\xi > \xi_T$, ϕ_0 exponentially vanishes, and the tail ϕ_T obeys a parabolic cylinder equation whose solution evolves as follows:

$$\phi_T(\xi,\varepsilon)=\phi_0(0,\varepsilon)\frac{e^{-\pi/(4\sqrt{\varepsilon})}}{\varepsilon^{1/4}\xi^{d/2+i/(2\sqrt{\varepsilon})}} \times \exp\left[\frac{i\sqrt{\varepsilon}\xi^2}{2}+i\varphi\right] \tag{7}$$

with the phase $\varphi=\{2\ln[1/(2\sqrt{\varepsilon})]-1\}/(4\sqrt{\varepsilon})+\pi/4$. Expression (7) of ϕ_T has been normalized in accordance with $\phi_0(\xi,\varepsilon)$. Primarily derived on the basis of the former semiclassical analysis, the twice logarithmic correction in the blowup rate (6) is usually expected to result from the approximated solution $2\sqrt{\varepsilon(\tau)}\approx a(\tau)\cong\pi/(\ln\tau)$ satisfying the following estimate:

$$\varepsilon_\tau=o(-\varepsilon/\tau)=-C\exp-\left[\frac{\pi}{2\sqrt{\varepsilon}}\right], \tag{8}$$

where C denotes a positive constant, which accounts for the mass transfer between the core ϕ_0 and the tail ϕ_T. Until now, finding this latter relation has constituted the major problem of a critical wave collapse, and often needed to invoke unclear arguments, as reviewed in Ref. [9]. Besides, the solutions resulting from these previous analytical investigations suffer from the important failure consisting in the fact that they exhibit a spatial logarithmic divergence in the integral $N\{\phi\}$ when computing the mass contribution associated with the tail (7). This divergence is in obvious contradiction with the mass conservation $N\{\psi\}=N\{\phi\}$ for which the integral N remains finite in space. As explained in Ref. [14], this failure follows from the approximation $\partial_\tau\phi=0$ made by most of the authors mentioned above. In [14], the spatial divergence of $N\{\phi\}$ was shown to result from the fact that the quasi-self-similar solution $\phi(\xi,\varepsilon)$ was supposed in the past to extend in the whole space domain: as will be recalled further on, such a solution is bounded from the top by a cutoff radius beyond which a *non*-self-similar remaining tail ensures the finiteness of the L^2 norm $N\{\phi\}$. In what follows, we detail this non-self-similar analysis applied to a critical wave collapse: by retaining the time derivative in Eq. (5), we precisely determine the non-self-similar contributions of the solution $\phi(\xi,\tau)$, which ensure a bounded integral $N\{\phi\}$. This analysis then allows us to derive a rapid and rigorous method used to find the blowup rate (6) whose dynamical stability near the singularity t_* is demonstrated.

Let us first solve the *nonstationary* linear problem (5) defined within the spatial range $\xi \geq \xi_0$, where $\xi_0=A\xi_T$ denotes some arbitrary point in the long tail domain, $A \gg 1$ being a numerical constant of order unity. In the following, the function $\varepsilon(\tau)$ is assumed to vary adiabatically in time [i.e., logarithmically, as suggested by the estimate (8)]. For further technical convenience, we set

$$\phi(\xi,\tau)=\left[\frac{x}{(2\sqrt{\varepsilon})^{1/2}}\right]^{(1-d)/2}\exp-i\left[\frac{\varepsilon_\tau x^2}{32\varepsilon\sqrt{\varepsilon}}\right]\phi'(x,\tau'), \tag{9}$$

where x and τ' denote new space and time variables defined by $x=\sqrt{2}\varepsilon^{1/4}\xi$ and $\tau'=\int_0^\tau 2\sqrt{\varepsilon(u)}du\cong\ln[1/g(\tau)]$, respectively. Substituting (9) into Eq. (5), we obtain

$$i\partial_{\tau'}\phi'+\partial_x^2\phi'+\frac{x^2}{4}\left[1-\frac{2}{\sqrt{\varepsilon}x^2}\right]\phi'=0, \tag{10}$$

where the inequalities $|a_\tau|/a^2 \ll 1 \ll x^2$ with $a \ll 1$ have been taken into account. In the space region $x \gg x_T\equiv(2/\sqrt{\varepsilon})^{1/2}$, the last term in $2/(\sqrt{\varepsilon}x^2)$ of Eq. (10) is an infinitesimal correction whose time variation with respect to $\tau'\approx 2\sqrt{\varepsilon}\tau$ can be ignored. Equation (10) has to be solved under the boundary conditions $\phi'(x_0(\tau'),\tau')=\phi'_T(x_0(\tau'),\tau')$ and $\partial_x\phi'(x\to\infty,\tau')=0$, where $x_0(\tau')=\sqrt{2}\varepsilon^{1/4}\xi_0$ and

$$\phi'_T(x)=Z(\varepsilon)\exp i[x^2/4-(\ln x)/2\sqrt{\varepsilon}]/\sqrt{x/x_T},$$

with

$$Z(\varepsilon)\equiv\exp\{-\pi/4\sqrt{\varepsilon}+i\pi/4+i[\ln(1/2\sqrt{\varepsilon})-1]/(4\sqrt{\varepsilon})\},$$

respectively, represent the point ξ_0 and the function ϕ_T through the substitution (9) in the new frame defined by x and τ'. Laplace transforming Eq. (10) and returning to the variables ξ and τ, we find that the solution $\phi(\xi,\tau)$ is constituted of two main components, namely, $\phi=\phi_{\rm I}+\phi_{\rm II}$. These two contributions can be computed within the basic approximation $|a_\tau| \ll a^2$; the main part $\phi_{\rm I}$ is given by

$$\phi_{\rm I}(\xi,\tau)=\phi_T(\xi,\varepsilon)\left[\frac{\tau\ln^3\tau}{\tilde\tau\ln^3\tilde\tau}\right]^{-\nu}H(\xi_{\max}-\xi), \tag{11a}$$

with

$$\tilde\tau\equiv\tau-\frac{1}{2\sqrt{\varepsilon}}\ln\left[\frac{\xi}{\xi_0}\right] \tag{11b}$$

and

$$\nu\equiv-\tfrac{1}{2}+\frac{i}{\pi}[A^2-\ln(2A)-\tfrac{1}{2}]. \tag{11c}$$

In Eq. (11a), $H(x)$ denotes the usual Heaviside function [$H(x)=1$ for $x>0$] and $\xi_{\max}$ is the cutoff radius defined by

$$\xi_{\max}=\xi_0/g(\tau). \tag{12}$$

The expression (11a) (whose similar form was recently derived by Malkin in Ref. [9]) only remains valid within the spatial range $\xi_T \ll \xi \ll \xi_{\max}$. In the limit $\xi\to\xi_{\max}$, $\phi_{\rm I}(\xi,\tau)$ is found to behave as follows:

$$\phi_{\rm I}(\xi,\tau)\approx\frac{\Gamma(\nu+1)\sin(\nu\pi/2)}{\nu\pi}\times\{\exp[-\nu\pi/(2\sqrt{\varepsilon})]\}(\varepsilon\xi_0^2)^{-\nu}\phi_T(\xi,\varepsilon)\ , \qquad (13)$$

which is of order $[g(\tau)]^{d/2}$ as ξ reaches the cutoff $\xi_{\max}$; in the latter expression, $\Gamma(x)$ simply denotes the usual Euler Γ function. Computed in the limit $\ln[1/g(\tau)]\approx 2\sqrt{\varepsilon}\tau \gg 1$, the remaining part $\phi_{\rm II}$ reads as

$$\phi_{\rm II}(\xi,\tau)=\frac{\Gamma(\nu+1)}{\pi\xi^{d/2}}\left[Z_1'(\tau)\{\theta+2\bar{\theta}\}\exp i(\sqrt{\varepsilon}\xi^2/2)+\frac{Z_2'(\tau)\exp -i(\sqrt{\varepsilon}\{\ln[1/g(\tau)]\}\xi^2/2)}{\xi^{2\nu+3/2}}\right]\ , \qquad (14a)$$

where $\bar{\theta}(\xi,\tau)$ represents the complex conjugate function of the following one:

$$\theta(\xi,\tau)\equiv[i\pi/2-\ln(\mu\xi/\xi_{\max})]^{-1}\times\exp\{-(i/2)[\sqrt{\varepsilon}\xi_0^2\ln(\mu\xi/\xi_{\max})+\pi(\nu+\tfrac{5}{4})]\}\ . \qquad (14b)$$

In Eqs. (14), the quantity μ is defined by $\mu\equiv\sqrt{2}\varepsilon^{1/4}\xi_0$, and the time-dependent factors $Z_1'(\tau)$ and $Z_2'(\tau)$ are respectively given by $Z_1'(\tau)\approx -i(\varepsilon^{1/4}/\sqrt{A})(\tau\ln^3\tau)^{-A^2/2}$, and by $Z_2'(\tau)\approx\{\ln[1/g(\tau)]\}^{-1}\varepsilon^{\nu+1/2}\exp i(\nu\pi/2)$. In the domain $\xi<\xi_{\max}$, the contribution $\phi_{\rm I}$ dominates the residual part $\phi_{\rm II}$; indeed, denoting by $M\{\phi\}=\int_{\xi_0}^{\xi_{\max}}|\phi|^2\xi^{d-1}d\xi$ the mass transferred into the solution at the point $\xi=\xi_0$, one can see that $M\{\phi_{\rm II}\}\approx a(\tau)\{\ln[1/g(\tau)]\}^{-2}$ is negligible in front of $M\{\phi_{\rm I}\}\approx(\pi^2/2)\{\ln\ln[1/g(\tau)]\}^{-2}$ in the asymptotic limit $g(\tau)\ll a(\tau)\ll 1$. Thus relations (11) simply express that the complete solution ϕ reduces to the quasi-self-similar solution $\phi(\xi,\varepsilon)$ in the spatial range $\xi\ll\xi_{\max}$ only. In the complementary domain $\xi>\xi_{\max}$, ϕ is given by the τ-dependent solution $\phi_{\rm II}$ whose spatial dependence ensures the L^2 convergence of the whole solution, which solves the spatial divergence problem that primarily occurred throughout the quasi-self-similar analysis.

Let us now recover the contraction rate (6) by taking the previous results into account: as originally introduced by Fraiman in [7], the central core is searched for under the form $\phi_0(\xi,\varepsilon)=c(\varepsilon)R_0(\xi)$ where the amplitude factor $c(\varepsilon)$ is assumed to be of order of unity for small nonzero values of ε. This function measures the difference between exact solution ϕ_0 defined for a finite τ and the stationary state R_0. Its functional dependence can be derived by inserting the solution $\phi=\phi_0+\phi_T$ into Eq. (5) and by multiplying it by $c^{-1}\xi^{d/2}\partial(\xi^{d/2}R_0)/\partial\xi$. Taking then the real part of the space-integrated result and using the relation $H\{R_0\}=0$ indeed yield the following estimate:

$$|c|^{4/d}=1+\varepsilon K\ , \qquad (15)$$

with $K=(2N\{\xi R_0\}/dN_0)$ and with $N_0=N\{R_0\}$. Expression (15), which depends linearly on ε, is valid in the domain $\varepsilon\ll 1$ as $\tau\to\infty$, and it describes the evolution of the quantity $|c|^2$ with an accuracy of the order of $\exp-[\pi/(4\sqrt{\varepsilon})]$. This linear dependence in ε justifies the perturbative methods originally based on a conjectured ε expansion of the solution ϕ (as, e.g., in [9]). Since the function $|c(\varepsilon)|^2$ tends to unity as $\varepsilon\to 0$ and since ξ_T tends to infinity in the same limit, one sees that for any bounded domain of ξ, the solution ϕ decreases asymptotically in time towards the core R_0. It is now necessary to derive the dynamical equation governing the time evolution of the function ε; the latter may be deduced from the previous functional dependence of $|c|^2$. Following the procedure used by Malkin in Ref. [9], we multiply Eq. (5) by $\xi^{d-1}\bar{\phi}$ and integrate the imaginary part of the result from zero to ξ to obtain the following continuity equation:

$$\int_0^\xi\frac{\partial}{\partial\tau}|\phi(\rho,\tau)|^2\rho^{d-1}d\rho=-2\xi^{d-1}|\phi(\xi,\tau)|^2\frac{\partial}{\partial\xi}\arg\phi(\xi,\tau)\ . \qquad (16)$$

By inserting ϕ as defined by expression (7) into the right-hand side of the continuity equation (16), one finds on the one hand that the latter reduces to $-2|\phi_0(0,\varepsilon)|^2\exp[-\pi/(2\sqrt{\varepsilon})]$ and thus no longer depends on ξ in the range $\xi\gg 1/\sqrt{\varepsilon}$, which corresponds to a uniform mass transfer in the tail domain. On the other hand, in this limit $\xi\to\infty$, one can identify the left-hand side (lhs) of (16) with $\partial_\tau N\{\phi\}$; however, assuming *a priori* $N\{\phi\}\approx N\{\phi_0\}$ would be incorrect when only using the approximation $\phi=\phi_0+\phi_T$, since $N\{\phi_T\}$ spatially diverges. The non-self-similar analysis performed above allows us to clear up this problem; indeed, it can be checked from the results (11) and (14) that the mass contribution of the residual tail satisfies the inequality $\int_{\xi_{\max}}^\infty|\phi|^2\xi^{d-1}d\xi\ll\int_0^{\xi_{\max}}|\phi|^2\xi^{d-1}d\xi$. Contributing to the latter integral, the mass $M\{\phi_{\rm I}\}$ can in turn be neglected as compared with the quantity $\int_0^{\xi_0}|\phi_0+\phi_T|^2\xi^{d-1}d\xi$ converging to $N\{\phi_0\}$ in the limit $g(\tau)\ll 1$. The mass $N\{\phi\}$ therefore reduces to $N\{\phi_0\}=|c|^2N_0$ as $\xi_{\max}$ tends to infinity. Applying this result to Eq. (16), whose lhs reads as $\partial_\tau N\{\phi_0\}$, finally leads to

$$\varepsilon_\tau=-C\exp[-\pi/(2\sqrt{\varepsilon})],\quad C\equiv\frac{2R_0^2(0)}{N\{\xi R_0\}}\ , \qquad (17a)$$

after taking the relations (15) and $|\phi_0(0,\varepsilon)|^2=|c(\varepsilon)|^2R_0^2(0)$ into account. Equation (17a) thus justifies the key estimate (8) from which the contraction rate (6) follows.

The tail solution (11), (13), and (14) together with the cutoff (12) can be viewed as corresponding to the following physical process: in the core region, $\phi(\xi,\tau)=c(\varepsilon)R_0(\xi)$ evolves towards $R_0(\xi)$ with $|c(\varepsilon)|$ continuously decreasing to unity as $\tau\to\infty$; this property is therefore associated to a mass input into the tail, from which the behavior $\xi_{\max}\to\infty$ follows. This mass transfer has now to be proved as corresponding to a stable dynamical process; it accounts for investigating the stability problem of the contraction rate (6) for $t \to t_*$, since the twice logarithmic correction in (6) results from this mass dynamics. This stability analysis is based on Eq. (17a), together with the following equations:

$$a_\tau = 4\varepsilon - a^2 , \tag{17b}$$

$$g_\tau = -ga , \tag{17c}$$

which constitute a complete dynamical system in $\mathbb{R}^3$ of the form $X_\tau = F(X)$ with the vector field $X(\tau) \equiv (\varepsilon(\tau), a(\tau), g(\tau))^T$. This system, hereafter denoted by S, exhibits a unique fixed point $X_\infty \equiv X(\infty)$ in which the vanishing components $\varepsilon(\infty) = a(\infty) = g(\infty) = 0$ respectively correspond to the asymptotic values taken by the positive functions $\varepsilon(\tau)$, $a(\tau)$, and $g(\tau)$ as τ reaches infinity. Because the scaling law $g(\tau) = g(\tau_0)\exp - \int_{\tau_0}^{\tau} a(s)ds$, where τ_0 refers to an initial time, surely converges towards its asymptotic zero value, S may be restrained to the nonlinear equations (17a) and (17b), which are characterized by an exponential decrease of ε_τ on the one hand, and by a parabolic functional dependence between $\varepsilon(\tau)$ and $a(\tau)$ on the other hand. Regarding now the quadratic dependence on a^2 of Eq. (17b), one can investigate the two distinct regions located around the parabola $\varepsilon = a^2/4$ that correspond to the different classes of initial data satisfying the inequalities $a^2(\tau_0) < 4\varepsilon(\tau_0)$ and $a^2(\tau_0) > 4\varepsilon(\tau_0)$, respectively. Since ε_τ is negative, the positive function $\varepsilon(\tau)$ always decreases from $\varepsilon(\tau_0)$ towards its stationary value $\varepsilon(\infty) = 0$. For initial data such as $a^2(\tau_0) > 4\varepsilon(\tau_0)$, the time derivative $\partial_\tau a$ in Eq. (17b) remains negative, so that the function $a(\tau)$ decreases to zero as $\tau \to \infty$. In the opposite region defined by $a^2(\tau_0) < 4\varepsilon(\tau_0)$, $\partial_\tau a$ is positive, and the function $a(\tau)$ increases until it crosses the parabola $\varepsilon = a^2/4$; afterwards one has $\partial_\tau a \leq 0$, so that $a(\tau)$ decreases to zero and

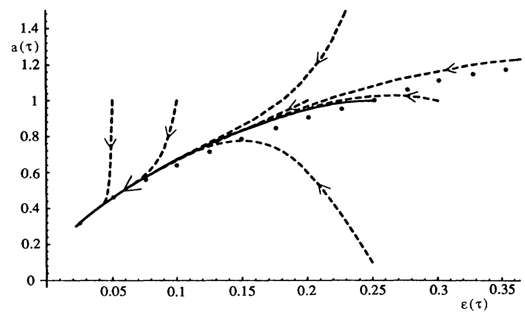

FIG. 1. Parametric plot of the time-dependent function $a(\tau)$ vs $\varepsilon(\tau)$ as described by Eqs. (17) with $C = 2$. For $\tau_0 = 0$, the solid curve corresponds to the initial datum $a(0) = 2\sqrt{\varepsilon(0)}$ and joins the parabola $\varepsilon = a^2/4$ (dotted line) as $\tau \to \infty$. Dashed lines represent trajectories integrated from the initial data $a(0) > 2\sqrt{\varepsilon(0)}$ and $a(0) < 2\sqrt{\varepsilon(0)}$. Arrows indicate the flow direction.

asymptotically behaves like $2\sqrt{\varepsilon(\tau)}$. In the limit $\varepsilon \to 0$, trajectories issued from initial data satisfying $a(\tau_0) = 2\sqrt{\varepsilon(\tau_0)}$ become close to the curve $\varepsilon = a^2/4$ following the relation $a^2 - 4\varepsilon = C(da/d\varepsilon)\exp[-\pi/(2\sqrt{\varepsilon})] \approx (C/\sqrt{\varepsilon})\exp[-\pi/(2\sqrt{\varepsilon})]$, as illustrated in Fig. 1. These arguments show that the fixed point X_∞ is stable. More precisely, integrating numerically Eqs. (17) as was done in Fig. 1 with $\tau_0 = 0$, reveals a very slow convergence of both the functions $a(\tau)$ and $\varepsilon(\tau)$ towards zero for times $\tau \gtrsim 100$. This finally endows the fixed point of S with a nature of logarithmic stability, as it could be guessed from the behavior $a(\tau) \approx \pi/(\ln\tau)$, which proves the stability of the blowup rate (6).

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